CONTINUUM MODELS OF HIGH DENSITY TRAFFIC FLOW

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ABSTRACT

Since high density traffic flow by definition involves a large number of vehicles, models of traffic flow which involve aggregate quantities such as density, flow rate and mean speed seem particularly relevant. In this paper, models following this approach are developed and explored in several stages, including the introduction of the conservation of cars equation, the Fundamental Road Diagram and representations of driver anticipation and reaction time. Particular attention is given to the study of stationary solutions and stability.

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1. Introduction

In recent years, attention to the problem of congested ...eway traffic has been growing. Among the more promising appro... .es, other than building more freeways, is the moderation of traffic congestion by on-ramp control [1]. The control procedures employed, however, are necessarily simple because the dynamics of dense freeway traffic are not yet well understood in the sense that acceptable mathematical models are available.

Several rather distinct approaches to traffic models have been taken. These are commonly grouped into categories as queueing, car-following, Boltzmann-like statistical and continuum models. (A rather complete critical bibliography of papers dealing with these models appears in reference [2].) Each of these approaches has its areas of applications; the continuum models seem particularly well suited to the modeling of dense freeway traffic.

In discussing the performance of a freeway, traffic engineers speak of the flow rate and density of vehicles. They are the variables with which continuum models deal directly. Unfortunately, continuum models have not been adequately developed since the work by Lighthill and Whitham [3]. In this paper, refinements to the model first proposed by Lighthill and Whitham are developed with emphasis given to the analysis of stationary traffic patterns and bottlenecks. The ultimate goal of the sort of analysis pursued here is to develop a mathematical model of freeway traffic which is sufficiently accurate to permit the intelligent design of effective means of control. The most serious

drawback of the approach taken here is that, to date, there has been no attempt to correlate the theory with data. The theory which is developed is justified solely on qualitative grounds, primarily on questions of stability.

In Section 2, the basis of the continuum approach is presented, including a derivation of the conservation of cars equation and the introduction of the Fundamental Road Diagram. In Section 3, driver anticipation is introduced into the model. The method of analysis of stationary solutions, bottlenecks and stability is presented in some detail here. In sections 4 and 5, these methods are applied to the study of models which include a representation of reaction time.

2. Continuum Models

The description of a large number of individual vehicles becomes difficult if only because of the fact of large numbers. This sort of problem is not uncommon in the physical sciences and is often overcome by changing one's point of view. Extreme examples of this occur in fluid and gas dynamics where one abandons any pretense of following the motion of individual molecules and

concentrates on average properties of large groups of molecules.
It is clear that the number of vehicles in any traffic situation is
not of the order of magnitude of number of molecules in a fluid,
but, nonetheless, it may be valuable to introduce new variables
which reflect average properties of relatively large numbers of
vehicles.

To illustrate the nature of this point of view, we shall
derive what we can properly call the conservation of cars equa-
tion. Suppose we station two observers on a road at points x_1
and x_2, x_2 being the forward point. Each observer counts
cars as they pass his station, and generate two functions $n_1(t)$
and $n_2(t)$, the accumulated number of cars counted. At the
same time we shall take motion pictures of the interval be-
tween these observers, counting the number of cars present
at any time, $n_3(t)$. Now it is clear that the number of cars

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in the interval changes over a period of time Δt according to the formula

$$n_3(t + \Delta t) - n_3(t) = [n_1(t + \Delta t) - n_1(t)] - [n_2(t + \Delta t) - n_2(t)].$$

We can introduce two average properties of the vehicle traffic. First the density of vehicles in this segment of road at time t can be defined as the number of vehicles per unit length of road, so that

$$\rho(t) = \frac{n_3(t)}{x_2 - x_1}.$$

Second, the flow rate over a period of time Δt, for example at location x_1, is

$$q(x_1) = \frac{n_1(t + \Delta t) - n_1(t)}{\Delta t}.$$

Our equation can then be written as

$$(x_2 - x_1)[\rho(t + \Delta t) - \rho(t)] = \Delta t[q(x_1) - q(x_2)]$$

or

$$\frac{\rho(t + \Delta t) - \rho(t)}{\Delta t} + \frac{[q(x_2) - q(x_1)]}{x_2 - x_1} = 0.$$

In the limit, where Δt and $x_2 - x_1$ are considered to be infintesimal, this reduces to

$$(1) \qquad \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0.$$

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It is clear that this last equation is really a sort of approximation since the quantities defined, ρ and q, lose their meaning (cannot be sensibly derived as indicated by observation) if Δt or $x_2 - x_1$ is too small. The preceding equation is exact, however.

This model of traffic is by no means complete at this point since nothing has been said about the dynamics of the traffic. The simplest model derives from what has been called the Fundamental Road Diagram [4], which specifies the relationship between the density and flow rate. This can easily be observed empirically [5] and depends on many factors: the specific road, the time of day, weather conditions, etc. One typically finds a relationship of the sort illustrated in Figure 1. It should be understood, of course, that the diagram represents the smoothed version of a great amount of experimental data.

The qualitative features depicted can be easily explained by a simple mathematical model. Driving experience has indicated that one travels more slowly when more cars are present on the road, i. e., when the density is higher. Letting u denote the space-mean speed of drivers in a segment of road, a simple model of this effect is

$$u = u_{max}(1 - \rho / \rho_{jam})$$

where ρ_{jam} is the maximum density existing when cars are at rest.

To relate average car speed to flow rate, consider a short

section of road through which vehicles are moving at constant speed and with fixed average spacing (so that the density remains constant). If the section has length ℓ and there are n vehicles, the average spacing is ℓ/n, and the density is n/ℓ. On the average, a car will pass the forward observer every Δt seconds, where

$$\Delta t = \left(\frac{\ell}{n}\right)\left(\frac{1}{u}\right) = \frac{1}{\rho u}$$

is the period of time that the vehicle takes to cover the spacing distance. The flow rate is just $1/\Delta t$, so that

$$q = \rho u .$$

Using the model for the speed-density relation above, we have

$$q = \rho u_{max}(1 - \rho/\rho_{jam}) .$$

This is the equation of a parabola and has the general shape of the Fundamental Road Diagram depicted. In particular, we note that if no cars are present, $\rho = 0$ and $q = 0$. If the vehicles are in a traffic jam, $\rho = \rho_{jam}$, the flow again vanishes. The maximum value of q, which for our model occurs at $\rho = \frac{1}{2}\rho_{jam}$ (and $u = \frac{1}{2}u_{max}$), is sometimes called the (maximum) capacity of the road. It should be clear that this is a very desirable operating point for a road.

The conservation of cars equation and the Fundamental Road Diagram provide a basis for a simple though crude qualitative description of traffic flow. One striking feature is the presence of discontinuous solutions representing zero thickness shock waves which correspond to the experience of drivers sharply applying their brakes in a sequence moving backward through traffic. A fairly complete discussion of this model has been presented by Lighthill and Whitham [3]. It is the purpose of the remainder of this paper to refine this model by including effects representing driver anticipation and reaction time.

3. An Anticipation Model

The avoidance of shock waves requires that we model driver anticipation in some way; for example, it is reasonable to suppose that the flow rate is reduced if the car spacing ahead is increasing. Designating the function represented in the Fundamental Road Diagram by $Q(\rho)$, this means

$$q = Q(\rho) - \nu \rho_x$$

ν a constant positive parameter [(subscripts denote partial differentiation)]. Now substituting into (1) one obtains

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(2) $$\rho_t + c \quad = \nu \rho_{xx}$$

where

$$c(\rho) \equiv \frac{dQ(\rho)}{d\rho}$$

Equation (2) represents only one possible refinement. In the following, we shall consider several possible models and discuss their validity on the basis of stability and ability to describe certain special traffic situations.

3.1 Analysis of Stationary Solutions

As soon as any refinements are made, e. g., those leading to (2), exact analytical solutions of any generality become impossible. However, one can successfully investigate stationary solutions, that is, solutions in which the density takes the form

$$\rho = \rho(x - Ut)$$

or

$$\rho = \rho(\xi), \quad \xi = x - Ut,$$

where U is a constant. Physically, solutions of this sort mean that an observer moving along a road at speed U would see an unchanging traffic pattern. Substituting this form into (2), one obtains

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$$-U\rho_\xi + c(\rho)\rho_\xi = \nu\rho_{\xi\xi} .$$

Not particularly that this reduces the number of independent variables to one. This can be integrated once to yield.

(3) $$-U\rho + Q(\rho) = \nu\rho_\xi + A ,$$

where A is a constant (recall that $c(\rho) = \dfrac{dQ}{d\rho}$). In fact A is the constant flow rate relative to an observer moving along the road with speed U. To see this, consider the conservation of cars equation

$$\rho_t + q_x = 0 .$$

In this stationary situation, $\rho_t = -U\rho_\xi$, $q_x = q_\xi$. Then, integrating,

$$q = U\rho + B,$$

B a constant. If $U = 0$, B is the flow past an observer fixed on the road. For $U > 0$, an observer moving with the traffic pattern observes a reduced flow rate moving past him, reduced in fact by an amount $U\rho$. Then $B = q - U\rho$ is the flow past this moving observer. Noting that

$$q = Q(\rho) - \nu\rho_x$$

we see that $B = Q(\rho) - \nu\rho_x - U\rho = A$, i. e., A has the interpretation suggested.

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Equation (3) can be separated as

$$\frac{\nu \, d\rho}{Q(\rho) - [U\rho + A]} = d\xi$$

For specific forms of $Q(\rho)$ this can be integrated to yield $\xi = \xi(\rho)$, then inverted to yield $\rho = \rho(\xi)$. As an example, consider

$$Q(\rho) = \rho u_{max}(1 - \rho/\rho_{jam})$$

Then

$$Q(\rho) - U\rho - A = -\rho^2 \frac{u_{max}}{\rho_{jam}} + \rho(u_{max} - U) - A$$

$$= \frac{u_{max}}{\rho_{jam}} (\rho - \rho_1)(\rho_2 - \rho)$$

where

$$\rho_{1,2} = \frac{1}{2} \rho_{jam}(1 - \frac{U}{u_{max}}) \mp \left[\frac{1}{4} \rho_{jam}^2 \left(1 - \frac{U}{u_{max}}\right)^2 - \frac{A\rho_{jam}}{u_{max}} \right]^{\frac{1}{2}}.$$

Note that we have chosen $\rho_1 \leq \rho_2$. Since $\rho_\xi \to \pm \infty$, we find from (3)

$$- U\rho_1 + Q(\rho_1) = A$$

$$- U\rho_2 + Q(\rho_2) = A$$

and from these A and U are determined:

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$$A \quad \frac{\rho_1 Q(\rho_2) - \rho_2 Q(\rho_1)}{\rho_1 - \rho_2} \quad ,$$

$$U = \frac{Q(\rho_1) - Q(\rho_2)}{\rho_1 - \rho_2} \quad .$$

The roots ρ_1 and ρ_2 have an elegant interpretation on the Fundamental

Road Diagram. (See Figure 2.) With $\rho_{\xi = 0}$, (3) can be written as

$$A + U\rho = Q(\rho) \cdot$$

The left hand side of this equation represents the straight-line depicted

as intersecting the Diagram curve $Q(\rho)$. The roots ρ_1 and ρ_2 are just

the densities at the intersections. Now

$$d\xi = \frac{\nu \rho_{jam}}{u_{max}} \frac{1}{(\rho_2 - \rho_1)} \left[\frac{1}{\rho - \rho_1} + \frac{1}{\rho_2 - \rho} \right] d\rho$$

and integrating

$$\xi + K = \frac{\nu \rho_{jam}}{u_{max}(\rho_2 - \rho_1)} \left[\log | \rho - \rho_1 | - \log | \rho_2 - \rho | \right]$$

$$= \frac{\nu \rho_{jam}}{u_{max}(\rho_2 - \rho_1)} \log \left| \frac{\rho - \rho_1}{\rho_2 - \rho} \right| \cdot$$

For $\rho_1 \leq \rho_2 \leq \rho \leq \rho_{jam}$ (see Figure 5)

$$\rho = \frac{\rho_1 - C\rho_2 \exp(\alpha \xi)}{1 - C \exp(\alpha \xi)}, \quad \xi \geq \xi* = \frac{1}{\alpha} \log\left(\frac{\rho_{jam} - \rho_1}{C(\rho_{jam} - \rho_2)}\right), \quad C > 0,$$

(and in particular $\rho \to \rho_2$ as $\xi \to \infty$, $\rho \to \rho_{jam}$ as $\xi \to \xi*$).

The solutions illustrated represent at least two familiar traffic situations. In the solution indicated to be valid for $\rho_1 \leq \rho \leq \rho_2$, set $\rho_2 = \rho_{jam}$. Then the solution represents the gradual slow down of vehicles to a jam, with the density far down the road being ρ_1.

The opposite situation, start up from a jam is represented by the solution indicated to be valid for $\rho_2 \leq \rho \leq \rho_{jam}$. There is some ambiguity in this case, however, since it does not seem reasonable to impose a condition on the density far up the road. Experience as a driver convinces one that car spacing changes rather abruptly at the head of a line of vehicles which are starting up from jam conditions. If we can estimate ρ_ξ at this point, we can generate a reasonable solution.

Consider three vehicles: the one that is next to start moving, the vehicle ahead and the vehicle behind. The center vehicle of the three is not moving because it takes a driver a finite time to respond to the fact that the car ahead is moving. Suppose this response time

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is τ and suppose vehicles accelerate uniformly (at first) with acceleration a. Then the vehicle ahead has moved a distance $\frac{1}{2} at^2$, $t \leq \tau$ before the center vehicle moves at all. The average separation is

$$\frac{1}{\tau} \int_0^\tau \frac{1}{2} at^2 \, dt = \frac{1}{6} a\tau^2 .$$

Adding a car length, ℓ_c, the density is on the average

$$\frac{1}{\frac{1}{6} a\tau^2 + \ell_c}$$

ahead of this center vehicle, while the density behind is $\frac{1}{\ell_c}$. The rate of change of density is then

$$\frac{\dfrac{1}{\frac{1}{6} a\tau^2 + \ell_c} - \dfrac{1}{\ell_c}}{\Delta \xi} .$$

For $\Delta \xi$ we shall use

$$\frac{1}{2} \left(\ell_c + \frac{1}{6} a\tau^2 + \ell_c \right)$$

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so that

$$\frac{\Delta \rho}{\Delta \xi} = \frac{\ell_c - \left[\ell_c + \frac{1}{6} a \tau^2 \right]}{\ell_c + \frac{1}{12} a \tau^2} = - \frac{1}{6} \frac{a \tau^2}{\ell_c + \frac{1}{12} a \tau^2}$$

For example, with $\tau = 1$ second, $a = 10$ ft/sec^2, $\ell_c = 15$ ft,

$$\frac{\Delta \rho}{\Delta \xi} = -.11 \approx - \frac{a \tau^2}{6 \ell_c} .$$

Now we can apply the above boundary condition, at $\xi = 0$, and again take $\rho_\xi \to 0$ as $\xi \to + \infty$. Then from (3)

$$- U \rho_{jam} = \frac{a \tau^2 \nu}{6 \ell_c} + A ,$$

$$- U \rho_\infty + Q(\rho_\infty) = A .$$

Note that U has been implicitly fixed. In fact every τ seconds a vehicle a distance ℓ_c back in the waiting queue starts to move. Thus

$$U = \frac{- \ell_c}{\tau} .$$

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The two equations deriving from the boundary condition then serve to fix A and the density far from the jam, ρ_∞. In fact the first equation defines A and ρ_∞ is determined as the root of the second equation. For example, for

$$Q(\rho) = \rho\, u_{max}(1 - \rho/\rho_{jam})$$

we have

$$\frac{\ell_c}{\tau}\,\rho_\infty + \rho_\infty u_{max}(1 - \rho_\infty/\rho_{jam}) = \frac{a\tau^2 v}{6\ell_c} + \frac{\ell_c}{\tau}\,\rho_{jam}\ .$$

Note that $\rho_{jam} = \dfrac{1}{\ell_c}$, so we find

$$\frac{\rho_\infty}{\rho_{jam}} = \frac{1}{2} + \frac{1}{2}\,\frac{\ell_c}{\tau\, u_{max}} + \left[-\frac{a\tau^2 v}{6u_{max}} + \frac{1}{4}\left(1 - \frac{\ell_c}{\tau\, u_{max}}\right)^2 \right]^{\frac{1}{2}}\ .$$

The upper sign has been taken in accordance with the nature of solutions as illustrated in figure (5) and valid for $\rho_\infty \leq \rho \leq \rho_{jam}$. The flow rate at infinity is given by Figure 5.

$$Q(\rho_\infty) = A + U\rho_\infty$$

$$= -U\rho_{jam} + \frac{a\tau^2 v}{6\ell_c} + U\rho_{jam}\left[\frac{1}{2}\left(1 + \frac{\ell_c}{\tau\, u_{max}}\right) + \left(\frac{-a\tau^2 v}{6u_{max}} + \frac{1}{4}\left[1 - \frac{\ell_c}{\tau\, u_{max}}\right]^2\right)^{\frac{1}{2}}\right]\ .$$

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Recalling that $U = -\dfrac{\ell_c}{\tau}$, $\rho_{jam} \ell_c = 1$, this reduces to

$$Q(\rho_\infty) = \frac{1}{2\tau}\left(1 - \frac{\ell_c}{\tau u_{max}}\right) + \frac{a\tau^2 \nu}{6\ell_c} - \frac{1}{\tau}\left[\frac{-a^2\tau\nu}{6u_{max}} + \frac{1}{4}\left(1 - \frac{\ell_c}{\tau u_{max}}\right)^2\right]^{\frac{1}{2}} .$$

The approach we have taken here to analyze start up from a jam can also be employed to study what happens when a speed limit restriction is removed at some fixed point on a road. Now, $U = 0$, and ρ_ξ can be estimated in a similar manner.

3.2 Stationary Bottlenecks

Variations in road conditions, for example a curve or a section of reduced speed limit, cause the Fundamental Road Diagram to vary. The situation for a bottleneck is illustrated in Figure 6. In terms of the specific model

$$Q(\rho) = \rho u_{max}(1 - \rho/\rho_{jam})$$

the variation can be most easily implemented by having $u_{max} = u_{max}(x)$.

Analytically, it will prove convenient to take u_{max} to be piecewise constant. So let us consider one specific problem. Suppose a bottleneck is located in the interval (O, L) and that $u_{max}(x) = u_b$

in the bottleneck, $u_{max}(x) = u$ elsewhere, $u_b \leq u$. The solutions we previously obtained are applicable here. We have only to determine and apply boundary conditions.

Since the location of the bottleneck is fixed to the road, a stationary traffic pattern implies $U = 0$. Then

$$Q(\rho) = \nu\rho_x + A .$$

If traffic conditions are characterized by fixed densities at $x = \pm\infty$, $\rho_x \to 0$ in either case. Then the flows at $\pm\infty$ must be the same and $A = Q_\infty$. The densities, however, need not be the same.

With A specified and $U = 0$, the roots ρ_1 and ρ_2 are determined for each of the three regions: upstream of the bottleneck, within the bottleneck, and downstream from the bottleneck. Specifically,

$$\rho_1 = \frac{1}{2}\rho_{jam} - \left[\frac{1}{4}\rho_{jam}^2 - \frac{Q_\infty \rho_{jam}}{u}\right]^{\frac{1}{2}} ,$$

$$\rho_2 = \frac{1}{2}\rho_{jam} + \left[\frac{1}{4}\rho_{jam}^2 - \frac{Q_\infty \rho_{jam}}{u}\right]^{\frac{1}{2}} ,$$

both upstream of and downstream from the bottleneck, while in the bottleneck

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If ρ is restricted by $\rho \leq \rho_1$, then

$$\xi + K = \frac{\nu \rho_{jam}}{u_{max}(\rho_2 - \rho_1)} \log \left(\frac{\rho_1 - \rho}{\rho_2 - \rho} \right)$$

and solving for ρ :

$$\rho = \frac{C\rho_2 \exp(\alpha \xi) - \rho_1}{C \exp(\alpha \xi) - 1} \; , \qquad \rho \leq \rho_1 \leq \rho_2 \; ,$$

where $\alpha = u_{max}(\rho_2 - \rho_1)/\nu \rho_{jam}$ and C is an arbitrary positive constant. Note that as $\xi \to + \infty$, $\rho \to \rho_2$ so that this solution cannot be valid for all positive ξ. And as $\xi \to - \infty$, $\rho \to \rho_1$. The complete behavior is sketched in Figure 3. The solution presented is then valid for $\xi \leq \xi_0 = \frac{1}{2}\log\left(\frac{\rho_1}{C\rho_2} \right)$.

In similar fashion, one finds for $\rho_1 \leq \rho \leq \rho_2$ (see Figure 4)

$$\rho = \frac{\rho_1 + C\rho_2 \exp(\alpha\xi)}{1 + C \exp(\alpha\xi)} \; , \qquad -\infty \leq \xi \leq \infty, \; C > 0 \; .$$

The constant C determines the value of ρ at $\xi = 0$.

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$$\rho_{1b} = \frac{1}{2}\rho_{jam} - \left[\frac{1}{4}\rho_{jam}^2 - \frac{Q_\infty \rho_{jam}}{u_b}\right]^{\frac{1}{2}},$$

$$\rho_{2b} = \frac{1}{2}\rho_{jam} + \left[\frac{1}{4}\rho_{jam}^2 - \frac{Q_\infty \rho_{jam}}{u_b}\right]^{\frac{1}{2}}.$$

Since Q_∞ has a maximum value of $\frac{1}{4}\rho_{jam}u_{max}$, ρ_1 and ρ_2 are always real. And ρ_{1b} and ρ_{2b} will be real if

$$u_b \geq \frac{4Q_\infty}{\rho_{jam}},$$

that is, if the capacity of the bottleneck (as determined by the maximum speed u_b) is greater than the flow at infinity. And in this case $\rho_1 < \rho_{1b} \leq \rho_{2b} < \rho_2$. Because of this, one finds that the only possible solutions are the ones illustrated in Figure 7. The density at $x = 0$ must be in the range $\rho_{1b} \leq \rho(o) \leq \rho_2$. The solution valid in the region $x < 0$ is of the form

$$\rho = \frac{\rho_1 + \rho_2 C_1 \exp(\alpha x)}{1 + C_1 \exp(\alpha x)},$$

with $C_1 > 0$. That valid in the region $x > L$ is of the same form (replace C_1 with C_3). In the bottleneck region, the solution is

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$$\rho = \frac{\rho_{1b} + \rho_{2b} C_2 \exp(\beta x)}{1 + C_2 \exp(\beta x)}$$

if $\rho(o) \geq \rho_{2b}$; or is

$$\rho = \frac{\rho_{1b} - \rho_{2b} C_2 \exp(\beta x)}{1 - C_2 \exp(\beta x)}$$

if $\rho(o) < \rho_{2b}$. In these

$$\alpha = \frac{(\rho_2 - \rho_1)u}{\nu \rho_{jam}}, \qquad \beta = \frac{(\rho_{2b} - \rho_{1b})u_b}{\nu \rho_{jam}}.$$

In the three regions there are the three constants C_1, C_2 and C_3 to be determined. Two equations for their determination are obtained by imposing continuity of density at the two boundaries:

at $x = 0$:
$$\frac{\rho_1 + \rho_2 C_1}{1 + C_1} = \frac{\rho_{1b} + \rho_{2b} C_2}{1 + C_2}$$

at $x = L$:
$$\frac{\rho_{1b} + \rho_{2b} C_2 D_2}{1 + C_2 D_2} = \frac{\rho_1 + \rho_2 C_3 D_3}{1 + C_3 D_3}$$

In these, $D_2 = \exp(\beta L)$, $D_3 = \exp(\alpha L)$. There does not seem to be any natural way of determining a unique solution by the imposition of a further condition.

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3.3 Stability of Stationary Solutions

Now that we have obtained stationary solutions, there arises the question of whether these solutions will ever be attained and maintained if the traffic pattern initially differs from the stationary pattern. To investigate this question, we shall use a perturbation analysis to examine small deviations from a given stationary solution.

The general time-dependent equation is

$$(4) \qquad \rho_t + C(\rho)\rho_x = \nu\rho_{xx} \; .$$

This, being a form of a diffusion equation, implies that the number of cars on the road remains constant (conservation of cars). To see this note that

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho\, dx = \int_{-\infty}^{\infty} \rho_t\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \Big[-Q(\rho) + \nu\rho_x \Big] dx$$

(Recall that $dQ/d\rho \equiv C(\rho)$.) If the density tends to constant values at either end of the road, integration by parts yields

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho\, dx = -Q(\rho)\Big|_{-\infty}^{\infty} \; ,$$

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i. e., the total number of cars increases with a rate equal to the net

inflow rate. If the flow rates at either end are identical, the number

of cars remains constant.

It is presumed that a stationary solution, $\rho^*(x - Ut)$, is known,

i. e.,

$$- U\rho^*_\xi + C(\rho^*)\rho^*_\xi = \nu\rho^*_{\xi\xi}$$

where $\xi = x - Ut$. Introducing

$$\rho' = \rho - \rho^*, \qquad \rho' = \rho'(\xi, t) ,$$

$$\rho'_t - U\rho'_\xi + C(\rho^* + \rho')\rho'_\xi + [C(\rho^* + \rho') - C(\rho^*)]\rho^*_\xi = \nu\rho'_{\xi\xi} .$$

It is instructive to consider the simplest case of uniform flow,

i. e. $\rho^* = $ constant. The equation for the disturbance then reduces to

$$\rho'_t - U\rho'_\xi + C(\rho^* + \rho')\rho'_\xi = \nu\rho'_{\xi\xi}$$

Linearizing this equation (expanding and keeping only terms of first

order in ρ') yields

$$\rho'_t - U\rho'_\xi + C(\rho^*)\rho'_\xi = \nu\rho'_{\xi\xi}$$

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In the uniform flow case U can be chosen arbitrarily. For convenience we take $U = C(\rho*)$ to obtain

$$\rho'_t = \nu \rho'_{\xi \xi} .$$

It is well known that solutions of this (heat) equation are always stable. Specifically, if the initial disturbance to the uniform flow is bounded, the disturbance dissipates with the traffic pattern returning to uniform flow.

An even stronger statement, based on the maximum-minimum principle [6], can be made. Given any initial traffic pattern, the extremes of density will never exceed the extremes in the initial pattern. This statement is not limited to small disturbances from uniform flow.

Consider (4) which governs the time dependent behavior of the traffic pattern. If the density is at an extremum, $\rho_x = 0$ so that at this point $\rho_t = \nu \rho_{xx}$. At a maximum of density, $\rho_{xx} \leq 0$, so that the density cannot be increasing at this point, i.e., $\rho_t \leq 0$. At a minimum of density, $\rho_{xx} \geq 0$ so that $\rho_t \geq 0$.

As a consequence, this traffic flow model is incapable of describing many interesting traffic situations in which small disturbances can lead to a breakdown of the traffic pattern, i.e., build up of large densities in certain segments of the road.

4. A Refined Model of Traffic Flow

In the previous section, the inadequacy of the simple model discussed there was demonstrated in the sense that the representation of some interesting and important traffic situations was preculded. The single characteristic lacking in that model which leads to this situation is a representation of delay in response to traffic conditions. In this section a model will be discussed which considers the fact that response to changing traffic conditions is not immediate.

In the model considered in the previous section, the flow rate actually attained was specified in terms of the density by

$$q = Q(\rho) - \nu \rho_x .$$

Here we shall distinguish the quantity on the right hand side as the desired flow rate and assume that the flow is adjusted by an amount proportional to the difference between actual and desired flow rates.

From the point of view of drivers, the natural means of adjustment is to accelerate or decelerate according to whether the current speed is less than or greater than that desired. Since flow rate divided by density is speed, the suggested model is

$$\frac{du}{dt} = - \alpha \left[\frac{q - Q(\rho) + \nu \rho_x}{\rho} \right] ,$$

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α a positive constant. In this, $\dfrac{du}{dt}$ is the substantial derivative of the speed, that is, the rate of change of speed as viewed by individual vehicles. In full

$$\frac{du}{dt} = u_t + uu_x .$$

Using $q = \rho u$, one obtains

(5) $\qquad q_t - \dfrac{q}{\rho}\rho_t + qq_x/\rho - q^2\rho_x/\rho^2 = -\alpha[\, q - Q(\rho) + \nu\rho_x \,].$

4.1 Stationary Solutions

First we shall consider the special case of unchanging traffic pattern fixed to the road. That is, $\rho_t = 0$. Then $q_x = 0$ follows from the conservation of cars equation. If the inflow rate is assumed constant with time, then $q_t = 0$ also. With these (5) reduces to

(6) $\qquad \rho_x = \dfrac{\alpha(q - Q(\rho))}{\dfrac{q^2}{\rho^2} - \alpha\nu} .$

The qualitative nature of the solutions can be seen with the aid of the Fundamental Road Diagram.

The quantities q, α and ν define what we shall term the critical density, ρ_c:

$$\rho_c = \frac{q}{\sqrt{\alpha\nu}} \quad .$$

The significance of ρ_c is that for $\rho < \rho_c$, the denominator of the right hand side of (6) is positive and negative for $\rho > \rho_c$. Note that ρ_c depends on the flow rate q.

As we shall see when some specific solutions are found, the location of ρ_c with respect to ρ_1 and ρ_2 as illustrated on the Fundamental Road Diagram above is quite important. Since ρ_c depends linearly on q, one easily sees that $\rho_1 \leq \rho_c \leq \rho_2$ for all q if and only if the straight line representing $\rho = q/\sqrt{\alpha\nu}$ intersects the Fundamental Road Diagram at the point of maximum flow. If the speed at maximum flow is denoted by u_{qmax}, one has the requirement $u_{qmax} = \sqrt{\alpha\nu}$. For the specific Diagram function

$$(7) \qquad Q(\rho) = \rho u_{max}(1 - \rho/\rho_{jam})$$

one finds

$$u_{qmax} = \frac{1}{2}u_{max} ,$$

so the requirement here is

$$u_{max} = 2\sqrt{\alpha\,\nu}$$

The slope of the Fundamental Road Diagram is the wave speed. At $\rho = 0$, we denote this by c_{max} as it is clear that the wave speed will attain a maximum here. Then $\rho_c < \rho_1$ for all q if and only if $\sqrt{\alpha\,\nu} > c_{max}$. For $Q(\rho)$ specified by (7) above, this becomes $\sqrt{\alpha\,\nu} > u_{max}$. Finally, in order that $\rho_2 < \rho_c$ for all q, one must have $\alpha\nu = 0$. Figure 9 summarizes this information.

The character of solutions can be deduced from (6) without a definite specification of $Q(\rho)$. One need only note the sign of ρ_x as a determined by (6) for ρ in certain ranges. For fixed q we shall discuss the various solutions according to the location of ρ_c.

Whenever $\rho > \rho_c$, the denominator in the expression for ρ_x, (6) is negative and positive otherwise. When $\rho < \rho_1$ or $\rho > \rho_2$ the numerator is positive; the numerator is negative when $\rho_1 < \rho < \rho_2$. These comments provide the basis for sketching all the possible solutions, illustrated in Figures 10 a, b, c.

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More general stationary patterns are obtained by assuming solutions of the form

$$\rho = \rho\,(x - Ut) = \rho\,(\xi)$$

The conservation of cars equation becomes

$$- U\rho_\xi + q_\xi = 0$$

from which

$$-U\rho + q = A \text{ (constant)}.$$

And then

$$(8) \qquad \rho_\xi = \frac{\alpha\,(A + U\rho - Q(\rho))}{\dfrac{(A + U\rho)^2}{\rho^2} - \alpha\nu}$$

A similar qualitative analysis can be done here.

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The roots ρ_1 and ρ_2 are now defined by the intersection of the Fundamental Road Diagram with the straight line $A + U\rho$. The numerator is positive for $\rho_1 < \rho < \rho_2$. The denominator is positive for

$$(A + U\rho)^2 > \alpha\nu\rho^2$$

or

$$\rho < \rho_c = \frac{A}{\sqrt{\alpha\nu} - U}$$

if we require $\sqrt{\alpha\nu} - U > 0$. The solutions are of the same qualitative form as illustrated for the $U = 0$ case.

4.2 Stationary Solutions for an Assumed Diagram Function

Here we shall develop the analytical stationary solutions for the Diagram function

$$Q(\rho) = \rho u_{max}(1 - \rho/\rho_{jam}).$$

Equation (8) can be written, in this case, as

$$(9) \qquad d\xi = \frac{\rho_{jam}(U^2 - \alpha\nu)}{\alpha u_{max}} \; \frac{(\rho - \rho_3)(\rho - \rho_4)}{\rho^2(\rho - \rho_1)(\rho - \rho_2)} \, d\rho$$

where

$$\rho_{1,2} = \tfrac{1}{2}\rho_{jam}\left(1 - \frac{U}{u_{max}}\right) \mp \left[\frac{1}{4}\rho_{jam}^2\left(1 - \frac{U}{u_{max}}\right)^2 - \frac{A\rho_{jam}}{u_{max}}\right]^{\frac{1}{2}},$$

$$\rho_3 = \rho_c = \frac{A}{-\sqrt{\alpha\nu} - U} \; ,$$

$$\rho_4 = \frac{-A}{\sqrt{\alpha\nu} + U} \; .$$

Equation (9) can be solved to yield the following solutions:

$$\xi + k = \frac{\rho_{jam}(U^2 - \alpha\nu)}{\alpha u_{max}}\left[a\log\rho - \frac{b}{\rho} + c\log|\rho - \rho_1| + d\log|\rho - \rho_2|\right]$$

where $\qquad b = \rho_3\rho_4/\rho_1\rho_2$

$$a = [b(\rho_1 + \rho_2) - (\rho_3 + \rho_4)]/\rho_1\rho_2$$

$$d = \frac{1 + a\rho_1 - b}{\rho_2 - \rho_1}$$

$$c = -a - d$$

and k is a constant which determines the absolute location of the traffic pattern on the road.

In the special case $U = 0$, these reduce to

$$x + k = \frac{- \nu \rho_{jam}}{u_{max}} \left[a \log \rho - \frac{b}{\rho} + c \log |\rho - \rho_1| + d \log |\rho - \rho_2| \right]$$

where now

$$b = \frac{- A u_{max}}{\alpha \nu \rho_{jam}} \;,$$

$$a = \frac{- u^2_{max}}{\alpha \nu \rho_{jam}} \;.$$

4.3 Stability

Stability of general stationary solutions is difficult to investigate so we shall limit our attention here to stability of uniform flow. To proceed we shall assume that the density and flow are perturbed from uniform flow conditions so that

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$$\rho = \rho_0 + \rho'$$

$$\rho = q_0 + q'$$

in which ρ_0 and q_0 are constants related by $q_0 = Q(\rho_0)$ and ρ' and q' are assumed to be small relative to ρ_0 and q_0, respectively. Introducing these expressions into the time dependent equation (5) and linearizing one obtains

$$\rho'_t + \rho'_x = 0$$

$$\frac{1}{\rho_0} q'_t - \frac{q_0}{\rho_0^2} \rho'_t + \frac{q_0 q'_x}{\rho_0^2} - \frac{q_0^2 \rho'_x}{\rho_0^3} = -\alpha \left[\frac{q' - c(\rho_0) \rho' + \nu \rho'_x}{\rho_0} \right].$$

If we assume an initial disturbance in flow rate of the form e^{ikx}, due to linearity, the time dependent solution will be of the form $e^{i\omega t} e^{ikx}$. The frequency ω is in general complex and is determined by the wave number k. The disturbance is stable if ω is confined to the upper half complex plane as in this case the time dependent factor will contain a damped exponential. Setting $q' = e^{i\omega t} e^{ikx}$, ρ' is deter-

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mined by the first perturbation equation to be

$$\rho' = \frac{-k}{\omega} e^{i\omega t} e^{ikx} .$$

Substituting these two expressions for ρ' and q' into the second equation one finds that ω is given by

$$\omega = \frac{-kq_0}{\rho_0} + \frac{1}{2} i\alpha \pm \left[\alpha k^2 \nu - \frac{1}{4} \alpha^2 + i\alpha k(c - q_0/\rho_0) \right]^{\frac{1}{2}}$$

From this, it follows that ω has nonnegative imaginary part if and only if

$$\alpha \nu \ge (c - q_0/\rho_0)^2 .$$

Note that if α is sufficiently large, uniform flow is always stable. This is in agreement with the result found for the simpler model discussed in section 3 — as it should be, for the refined model here reduces to the previous model when $\alpha \to \infty$.

For the parabolic Diagram function, this condition becomes

$$\frac{\sqrt{\alpha \nu}}{u_{max}} \ge \frac{\rho}{\rho_{jam}} .$$

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There is a second "critical density" which determines stability defined
by

$$\rho_s = \rho_{jam} \frac{\sqrt{\alpha \nu}}{u_{max}}$$

For $\rho < \rho_s$, uniform flow is stable and unstable otherwise. If
$\sqrt{\alpha \nu}/u_{max} \geq 1$, any uniform flow is stable. For $0 \leq \frac{\sqrt{\alpha \nu}}{u_{max}} \leq 1$ only
certain uniform flows are stable. One can see that for sufficiently
small density (for high enough speed) the flow is stable but that high
density (low speed) flows will be unstable.

4.4 Stationary Bottlenecks

One traffic situation in which stationary solutions are important
is the presence of a bottleneck. We shall represent the bottle neck by
a region in which the Diagram function has reduced values from those
for the Diagram function pertaining to the remainder of the road. The
situation is illustrated below. $Q_b(\rho)$ is the Diagram function pertain-
ing to the bottleneck.

The various possible solutions which are constructed from the

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stationary solutions sketched in Section 4.1 are illustrated. The location of the critical density ρ_c is seen to have a significant effect on the nature of the solutions.

Any uniform flow representing a portion of the flow may be stable or unstable according to whether $\rho < \rho_s$ or $\rho > \rho_s$ independent of location of ρ_c. (The values of ρ_s and ρ_c are related to one another through common dependencies, however.)

The solutions represented by Figures 13b and 13f are particularly interesting in that the existence of the solutions represented by the lowest curves in each of these figures depends on the parameters of the traffic situation. For example, with other parameters fixed, if the bottleneck exceeds a certain length, these solutions are no longer possible. Or viewed from another point of view, with other parameters fixed, these solutions are not possible if the flow rate exceeds a fixed quantity. The only possible solution then is the one in which traffic exits from the bottleneck tending toward the high density (low speed) condition. In other words, as the inflow rate is increased, a point is reached at which the high speed traffic pattern "breaks down". When a specific Diagram function is specified, the traffic conditions for breakdown can be determined.

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5. A Second Refined Model of Traffic Flow

The model discussed in this section is a refinement of that discussed in the previous section in that the anticipation parameter ν is replaced by βu. This reflects the fact that anticipation is usually based on the car ahead so that as the speed decreases and headway (car spacing) decreases, a driver is less able to anticipate density changes. The model is then

$$\frac{du}{dt} = -\alpha \left[\frac{q - Q(\rho) + \beta u \rho_x}{\rho} \right] .$$

Using $u = q/\rho$, this can be rewritten in the form

$$(10) \qquad q_t - \frac{q}{\rho} \rho_t + q q_x / \rho - q^2 \rho_x / \rho^2 = -\alpha \left[q - Q(\rho) + \beta \frac{q}{\rho} \rho_x \right]$$

5.1 Stationary Solutions

Making the usual assumptions, one obtains

$$\rho_x = \frac{\alpha(q - Q(\rho))}{\dfrac{q^2}{\rho^2} - \alpha \beta \dfrac{q}{\rho}} = \frac{\alpha \rho^2 (q - Q(\rho))}{q(q - \alpha \beta \rho)}$$

The combination of parameters $\alpha\beta$ clearly represents a critical speed. When drivers proceed at a speed less than $u_c = \alpha\beta$, the nature of the flow changes. The qualitative nature of the stationary solutions here is identical to that for the model discussed in the previous model.

5.2 Stability

The stability analysis in Section 4.3 can be directly employed if ν is replaced by $\beta u_0 = \beta q_0 / \rho_0$. Uniform flow is stable if

$$\alpha\beta \frac{q_0}{\rho_0} \geq \left(c - \frac{q_0}{\rho_0} \right)^2 .$$

For the parabolic Diagram function, this becomes

$$\alpha\beta \geq \frac{u_{max} \rho_0^2}{\rho_{jam}^2 \left(1 - \rho_0 / \rho_{jam} \right)} .$$

It is clearly impossible for this inequality to be satisfied independent of ρ_0. For ρ_0 sufficiently near ρ_{jam}, uniform flow is unstable regardless of the magnitude of the sensitivity and anticipation parameters α and β.

It should be interesting to examine the nature of solutions for those ranges of density for which uniform flow is unstable. One possible solution may be a representation of "stop-and-go" traffic. Newell [7] has represented this phenomena by a "two-state" theory.

6. Summary and Conclusion

The continum models developed here are an extension of previous such models in that the effect of driver anticipation and reaction time have been included. Without a representation of reaction time, continuum models appear not to be capable of describing many interesting traffic situations. When such a representation is included in the model, a critical density arises (in section 5, a critical speed arises) above which uniform flow is no longer stable. This is probably the most outstanding feature of the model and accords with driving experience on congested freeways.

The study of stationary solutions pertaining to bottlenecks has indicated that traffic parameters determine whether a favorable passage of vehicles through the bottleneck is possible. In particular, with other parameters fixed, there is a critical flow rate above which the traffic conditions break down in the sense that only a high density state for vehicles passing through the bottleneck is possible.

At several points in the analysis a specific flow density relationship has been assumed. The form chosen is not intended as an accurate description of traffic, but rather was chosen for analytic convenience to illustrate the methods. The significant qualitative features discussed are independent of the specific flow-density relationship.

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REFERENCES

1. Newman, L., Dunnett, A., and Meis, G., "Freeway Ramp Control - What it Can and Cannot Do", to appear in Traffic Engineering.

2. "Analytical Models of Unidirectional Multi-Lane Traffic Flow: A Survey of the Literature", System Development Corporation TM-3791/001/02, 31 January 1969.

3. Lighthill, M. J., and Whitham, G. B., "On Kinematic Waves II. A Theory of Traffic Flow on Long Crowded Roads", Proc. Roy. Soc. London, Series A, vol. 229 (May 1955), pp. 317-345.

4. Haight, F. A., Mathematical Theories of Traffic Flow, Academic Press, New York, 1963.

5. Ayanian, H., "A Field Procedure for Determining Effects of Land Closures", Freeway Operation Department District 7, California Division of Highways, May 1969.

6. Hellwig, G., Partial Differential Equations, Blaisdell, New York, 1960.

7. Newell, G. F., "Theories of Instability in Dense Highway Traffic", J. Op. Res. Soc. Japan, No. 5, 1962, pp. 9-54.

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Figure 1.

FUNDAMENTAL ROAD DIAGRAM

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Figure 2.

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Figure 3.

Figure 4.

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Figure 5.

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Figure 6.

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Figure 7.

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Figure 8.

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$\dfrac{\sqrt{\alpha\nu}}{u_{max}} = 0$	$\rho_c > \rho_2$ for all q
$0 < \dfrac{\sqrt{\alpha\nu}}{u_{max}} < \tfrac{1}{2}$	$\rho_1 < \rho_c$ for all q $\rho_c \gtrless \rho_2$ depending on q
$\dfrac{\sqrt{\alpha\nu}}{u_{max}} = \tfrac{1}{2}$	$\rho_1 \leq \rho_c \leq \rho_2$ for all q
$\tfrac{1}{2} < \dfrac{\sqrt{\alpha\nu}}{u_{max}} < 1$	$\rho_c < \rho_2$ for all q $\rho_c \lessgtr \rho_1$ depending on q
$1 < \dfrac{\sqrt{\alpha\nu}}{u_{max}}$	$\rho_c \leq \rho_1$ for all q

Figure 9.

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$\rho_c < \rho_1$:

ρ_{jam}

ρ_2

ρ_1

ρ_c

O

$x \rightarrow$

Figure 10a

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$\rho_1 < \rho_c < \rho_2:$

ρ_{jam}

ρ_2

ρ_c

ρ_1

O

$x \rightarrow$
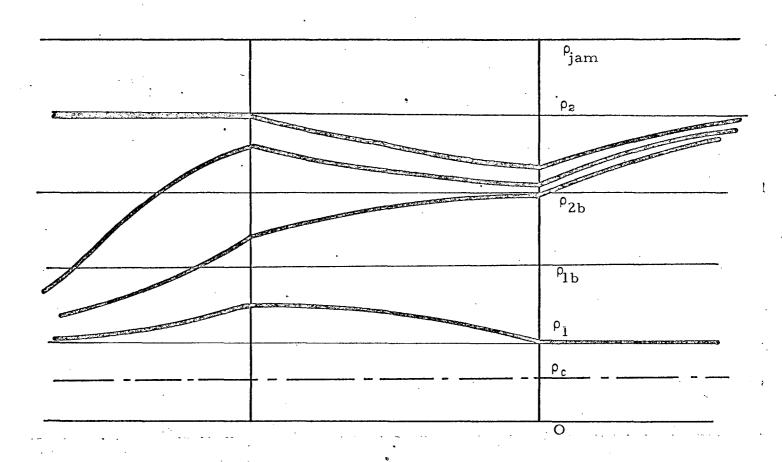
Figure 10b

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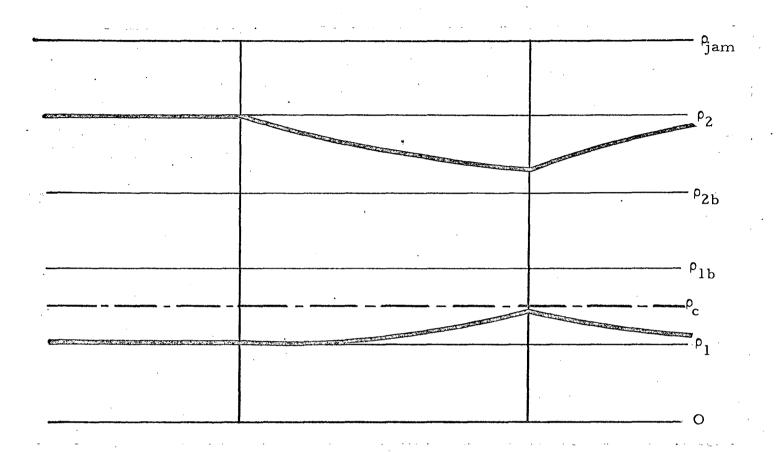Figure 10c

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Figure 11

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Figure 12.

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Figure 13a

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Figure 13b

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Figure 13c

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Figure 13d

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Figure 13 e

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Figure 13f

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